Discrete Mathematics

UNIT-2

Proposition

- A proposition is a declarative statement that is either true or false.
- It is denoted by a capital letter, such as P, Q, R, etc.

Example Question 1: Determine whether the following statement is a proposition: "The sky is blue."

Answer 1: Yes, it is a proposition.

Example Question 2: Is the statement "2 + 2 = 5" a proposition?

Answer 2: Yes, it is a proposition.

Example Question 3: Identify the proposition in the following statement: "All dogs are mammals."

Answer 3: The proposition is "All dogs are mammals."

Example Question 4: Determine whether the statement "It is raining outside" is a proposition.

Answer 4: Yes, it is a proposition.

Example Question 5: Is the statement "Mathematics is fun" a proposition?

Answer 5: Yes, it is a proposition.

Conjunction

- Conjunction combines two propositions using the word "and."
- It is denoted by the symbol $\wedge$ (logical AND).
- The resulting proposition is true only if both individual propositions are true.

Example Question 1: Evaluate the truth value of P ∧ Q, given P is true and Q is false.

Answer 1: P ∧ Q is false.

Example Question 2: If P represents "It is sunny" and Q represents "The temperature is hot," write the conjunction for "It is sunny and the temperature is hot."

Answer 2: P ∧ Q

Example Question 3: True or false: P ∧ P is always true for any proposition P.

Answer 3: True.

Example Question 4: Determine the truth value of (P ∧ Q) ∧ R if P, Q, and R are all true.

Answer 4: (P ∧ Q) ∧ R is true.

Example Question 5: Is the conjunction "P ∧ ¬Q" equivalent to "¬Q ∧ P"?

Answer 5: Yes, they are logically equivalent.

Topic: Disjunction

- Disjunction combines two propositions using the word "or."
- It is denoted by the symbol ∨ (logical OR).
- The resulting proposition is true if at least one of the individual propositions is true.

Example Question 1: Evaluate the truth value of P ∨ Q, given P is false and Q is true.

Answer 1: P ∨ Q is true.

Example Question 2: If P represents "It is raining" and Q represents "I have an umbrella," write the disjunction for "It is raining or I have an umbrella."

Answer 2: P ∨ Q

Example Question 3: True or false: P ∨ P is always true for any proposition P.

Answer 3: True.

Example Question 4: Determine the truth value of (P ∨ Q) ∨ R if P, Q, and R are all false.

Answer 4: (P ∨ Q) ∨ R is false.

Example Question 5: Is the disjunction "P ∨ ¬Q" equivalent to "¬Q ∨ P"?

Answer 5: Yes, they are logically equivalent.

Negation

- Negation changes the truth value of a proposition.
- It is denoted by the symbol ¬ (logical NOT).
- If the original proposition is true, its negation is false, and vice versa.

Example Question 1: Find the negation of the proposition P: "The moon is made of cheese."

Answer 1: The negation of P is ¬P: "The moon is not made of cheese."

Example Question 2: True or false: ¬(¬P) is equivalent to P for any proposition P.

Answer 2: True.

Example Question 3: If P represents "It is sunny," what is the negation of P?

Answer 3: The negation of P is ¬P: "It is not sunny."

Example Question 4: Determine the truth value of ¬(P ∨ Q) if P and Q are both true.

Answer 4: ¬(P ∨ Q) is false.

Example Question 5: Is the negation of "P ∧ Q" equivalent to "¬P ∨ ¬Q"?

Answer 5: Yes, they are logically equivalent.

Conditional Statements

- Conditional statements express an implication between two propositions.
- It is denoted by the symbol $\rightarrow$ (conditional or implication).
- The resulting proposition is false only if the antecedent (the first proposition) is true and the consequent (the second proposition) is false.

Example Question 1: Determine the truth value of $P \rightarrow Q$, given P is false and Q is true.

Answer 1: $P \rightarrow Q$ is true.

Example Question 2: If P represents "It is raining" and Q represents "The ground is wet," write the conditional statement for "If it is raining, then the ground is wet."

Answer 2: $P \rightarrow Q$

Example Question 3: True or false: $P \rightarrow P$ is always true for any proposition P.

Answer 3: True.

Example Question 4: Determine the truth value of $(P \rightarrow Q) \rightarrow R$ if P, Q, and R are all true.

Answer 4: $(P \rightarrow Q) \rightarrow R$ is true.

Example Question 5: Is the conditional statement "$P \rightarrow \neg Q$" equivalent to "$\neg Q \rightarrow P$"?

Answer 5: No, they are not logically equivalent.

Bi-conditional Statements

- Bi-conditional statements express a two-way implication between two propositions.
- It is denoted by the symbol $\leftrightarrow$ (bi-conditional or equivalence).
- The resulting proposition is true if both individual propositions have the same truth value.

Example Question 1: Evaluate the truth value of $P \leftrightarrow Q$, given P is true and Q is true.

Answer 1: $P \leftrightarrow Q$ is true.

Example Question 2: If P represents "I am happy" and Q represents "I am smiling," write the bi-conditional statement for "I am happy if and only if I am smiling."

Answer 2: $P \leftrightarrow Q$

Example Question 3: True or false: $P \leftrightarrow P$ is always true for any proposition P.

Answer 3: True.

Example Question 4: Determine the truth value of $(P \leftrightarrow Q) \leftrightarrow R$ if P, Q, and R are all false.

Answer 4: $(P \leftrightarrow Q) \leftrightarrow R$ is true.

Example Question 5: Is the bi-conditional statement "$P \leftrightarrow \neg Q$" equivalent to "$\neg Q \leftrightarrow P$"?

Answer 5: Yes, they are logically equivalent.

<u>**Compound Proposition**</u>

- Compound propositions are formed by combining multiple logical operators and propositions.
- They can be evaluated using truth tables to determine their truth values.

Example Question 1: Evaluate the truth value of $(P \vee Q) \wedge \neg R$, given P is true, Q is false, and R is true.

Answer 1: $(P \vee Q) \wedge \neg R$ is false.

Example Question 2: Write the compound proposition for "The sun is shining and it is not raining."

Answer 2: $P \wedge \neg Q$

Example Question 3: True or false: (P ∧ Q) ∨ R is equivalent to (P ∨ R) ∧ (Q ∨ R).

Answer 3: False.

Example Question 4: Determine the truth value of ¬(P ∧ Q) ∨ (P ∨ R) if P is true, Q is true, and R is false.

Answer 4: ¬(P ∧ Q) ∨ (P ∨ R) is true.

Example Question 5: Simplify the compound proposition (P ∨ Q) ∧ (¬P ∨ R).

Answer 5: (P ∨ Q) ∧ (¬P ∨ R) simplifies to Q ∨ (¬P ∧ R).

Truth Tables

- Truth tables display all possible combinations of truth values for a compound proposition.
- Each row represents a specific combination of truth values for the individual propositions.
- The last column represents the resulting truth value of the compound proposition.

Example Question 1: Construct a truth table for P ∧ Q.

Answer 1:

P	**Q**	**P ∧ Q**
T	T	T
T	F	F

F	T	F
F	F	F

Example Question 2: Create a truth table for P ∨ Q.

Answer 2:

P	**Q**	**P ∨ Q**
T	T	T
T	F	T
F	T	T
F	F	F

Example Question 3: Find the truth values for the compound proposition (P ∧ Q) → R, given P is true, Q is true, and R is false.

Answer 3:

P	**Q**	**R**	**(P ∧ Q) → R**

T	T	F	F

Example Question 4: Evaluate the truth value of the compound proposition $(P \rightarrow Q) \lor R$, given P is false, Q is true, and R is true.

Answer 4:

P	**Q**	**R**	**$(P \rightarrow Q) \lor R$**
F	T	T	T

Example Question 5: Construct a truth table for $\neg(P \lor Q)$.

Answer 5:

P	**Q**	**$\neg(P \lor Q)$**
T	T	F
T	F	F
F	T	F
F	F	T

Tautologies and Contradictions

- Tautologies are compound propositions that are always true, regardless of the truth values of the individual propositions.
- Contradictions are compound propositions that are always false.

Example Question 1: Determine whether the compound proposition $P \lor \neg P$ is a tautology.

Answer 1: Yes, $P \lor \neg P$ is a tautology.

Example Question 2: Is the compound proposition $(P \land Q) \lor (\neg P \land \neg Q)$ a tautology?

Answer 2: Yes, it is a tautology.

Example Question 3: Identify the contradiction in the compound proposition $(P \land \neg P) \lor (Q \land \neg Q)$.

Answer 3: The contradiction is $(P \land \neg P)$.

Example Question 4: Determine whether the compound proposition $(P \lor Q) \land (P \land Q)$ is a contradiction.

Answer 4: No, it is not a contradiction.

Example Question 5: Is the compound proposition $(P \leftrightarrow Q) \lor (\neg P \land \neg Q)$ a tautology?

Answer 5: No, it is not a tautology.

Logical Equivalence

- Two compound propositions are logically equivalent if they have the same truth values for all possible combinations of truth values for the individual propositions.
- Equivalence is denoted by the symbol $\equiv$.

Example Question 1: Determine whether the compound propositions $P \land (Q \lor R)$ and $(P \land Q) \lor (P \land R)$ are logically equivalent.

Answer 1: Yes, they are logically equivalent.

Example Question 2: Are the compound propositions $P \rightarrow Q$ and $\neg P \lor Q$ logically equivalent?

Answer 2: Yes, they are logically equivalent.

Example Question 3: True or false: $P \land Q$ is logically equivalent to $Q \land P$ for any propositions P and Q.

Answer 3: True.

Example Question 4: Determine whether the compound propositions $(P \rightarrow Q) \rightarrow R$ and $P \rightarrow (Q \rightarrow R)$ are logically equivalent.

Answer 4: Yes, they are logically equivalent.

Example Question 5: Are the compound propositions $(P \land Q) \lor R$ and $P \land (Q \lor R)$ logically equivalent?

Answer 5: No, they are not logically equivalent.

De Morgan's Law

De Morgan's Law is a fundamental concept in propositional logic that describes the relationship between negation and conjunction/disjunction. It provides a way to express the negation of a compound proposition in terms of its individual propositions. De Morgan's Law is composed of two parts: the negation of a conjunction and the negation of a disjunction.

Negation of a Conjunction:

The negation of a conjunction $(P \land Q)$ is equivalent to the disjunction of the negations of the individual propositions: $\neg P \lor \neg Q$.

Example 1: Apply De Morgan's Law to $\neg(P \land Q)$.

Solution 1: $\neg(P \land Q)$ is equivalent to $(\neg P \lor \neg Q)$.

Example 2: Use De Morgan's Law to rewrite $\neg(A \land B \land C)$.

Solution 2: $\neg(A \land B \land C)$ is equivalent to $(\neg A \lor \neg B \lor \neg C)$.

Negation of a Disjunction:

The negation of a disjunction $(P \lor Q)$ is equivalent to the conjunction of the negations of the individual propositions: $\neg P \land \neg Q$.

Example 3: Rewrite $\neg(P \lor Q)$ using De Morgan's Law.

Solution 3: $\neg(P \lor Q)$ is equivalent to $(\neg P \land \neg Q)$.

Example 4: Apply De Morgan's Law to $\neg(A \lor B \lor C)$.

Solution 4: $\neg(A \lor B \lor C)$ is equivalent to $(\neg A \land \neg B \land \neg C)$.

De Morgan's Law can also be extended to more complex compound propositions by applying the negation to the entire proposition and distributing it to the individual propositions.

Example 5: Use De Morgan's Law to rewrite $\neg((P \land Q) \lor R)$.

Solution 5: $\neg((P \land Q) \lor R)$ can be rewritten as $(\neg P \lor \neg Q) \land \neg R$ by applying De Morgan's Law to the conjunction and negating the disjunction.

In summary, De Morgan's Law provides a way to express the negation of conjunction as a disjunction of negations, and the negation of a disjunction as a conjunction of negations. It is a powerful tool in simplifying and manipulating compound propositions in propositional logic.

Example Question 6: Apply De Morgan's Law to $\neg(P \land Q)$.

Answer 1: $\neg(P \land Q)$ is equivalent to $(\neg P \lor \neg Q)$.

Quantifiers

- Quantifiers are used to express the scope of variables in mathematical statements.
- Universal quantifier ($\forall$): Represents "for all" or "for every."
- Existential quantifier ($\exists$): Represents "there exists" or "there is at least one."

Example Question 1: Express the statement "Every student passed the exam" using a quantifier.

Answer 1: $\forall x\, (\text{Student}(x) \rightarrow \text{PassedExam}(x))$

Example Question 2: Rewrite the statement "There exists a prime number less than 10" using a quantifier.

Answer 2: $\exists x\, (\text{Prime}(x) \land x < 10)$

Example Question 3: True or false: $(\exists x)P(x)$ is equivalent to $(\neg\forall x)\neg P(x)$.

Answer 3: True.

Example Question 4: Express the negation of "For every natural number, there exists a greater natural number."

Answer 4: $\neg(\forall x)\exists y\, (\text{NaturalNumber}(x) \rightarrow (\text{GreaterNumber}(y) \land y > x))$

Example Question 5: Rewrite the statement "There is a real number that is not positive" using a quantifier.

Answer 5: $\exists x\, (\text{RealNumber}(x) \land \neg\text{Positive}(x))$

Valid Arguments

- Valid arguments are deductive reasoning patterns where the conclusion logically follows from the premises.
- To determine validity, truth tables, logical equivalences, and inference rules can be used.

Example Question 1: Determine the validity of the argument: "If it is raining, then the ground is wet. The ground is wet. Therefore, it is raining."

Answer 1: The argument is invalid.

Example Question 2: Evaluate the validity of the argument: "All dogs have four legs. Fido is a dog. Therefore, Fido has four legs."

Answer 2: The argument is valid.

Example Question 3: True or false: If an argument is valid, it guarantees that the conclusion is true.

Answer 3: True.

Example Question 4: Determine the validity of the argument: "If it is Monday, then John has a meeting. It is Monday. Therefore, John has a meeting."

Answer 4: The argument is valid.

Example Question 5: Evaluate the validity of the argument: "If a shape has four equal sides, then it is a square. This shape is a square. Therefore, it has four equal sides."

Answer 5: The argument is valid.

Rules of Inference

- Rules of inference are logical rules used to derive conclusions from premises in valid arguments.

1. Modus Ponens:
    - If we have a conditional statement of the form "If P, then Q" and we know that P is true, we can conclude that Q is also true.

- Symbolically: $(P \rightarrow Q), P \vdash Q$.

2. Modus Tollens:
    - If we have a conditional statement of the form "If P, then Q" and we know that Q is false, we can conclude that P is also false.
    - Symbolically: $(P \rightarrow Q), \neg Q \vdash \neg P$.
3. Hypothetical Syllogism:
    - If we have two conditional statements "If P, then Q" and "If Q, then R", we can conclude that "If P, then R".
    - Symbolically: $(P \rightarrow Q), (Q \rightarrow R) \vdash (P \rightarrow R)$.
4. Disjunctive Syllogism:
    - If we have a disjunction statement "$P \vee Q$" and we know that one of the disjuncts is false ($\neg P$ or $\neg Q$), we can conclude the other disjunct.
    - Symbolically: $(P \vee Q), \neg P \vdash Q$ or $(P \vee Q), \neg Q \vdash P$.
5. Addition:
    - If we have a proposition P, we can conclude the disjunction statement "$P \vee Q$" for any proposition Q.
    - Symbolically: $P \vdash (P \vee Q)$ or $P \vdash (Q \vee P)$.

Example 1: Modus Ponens

Premise 1: If it is raining, then the ground is wet.

Premise 2: It is raining.

Conclusion: Therefore, the ground is wet.

Example 2: Modus Tollens

Premise 1: If it is a mammal, then it gives birth to live young.

Premise 2: It does not give birth to live young.

Conclusion: Therefore, it is not a mammal.

Example 3: Hypothetical Syllogism

Premise 1: If it is a dog, then it has fur.

Premise 2: If it has fur, then it is warm-blooded.

Conclusion: Therefore, if it is a dog, then it is warm-blooded.

Example 4: Disjunctive Syllogism

Premise 1: It is either sunny or rainy.

Premise 2: It is not sunny.

Conclusion: Therefore, it is rainy.

Example 5: Addition

Premise: I am a student.

Conclusion: Therefore, I am a student or I am a teacher.

Note: Each example is worth 2 marks.

Regenerate response